

# Detecting knot invertibility

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## Abstract

We discuss the consequences of the possibility that Vassiliev invariants do not detect knot invertibility as well as the fact that quantum Lie group invariants are known not to do so. On the other hand, finite group invariants, such as the set of homomorphisms from the knot group to  $M_{11}$ , can detect knot invertibility. For many natural classes of knot invariants, including Vassiliev invariants and quantum Lie group invariants, we can conclude that the invariants either distinguish all oriented knots, or there exist prime, unoriented knots which they do not distinguish.

Not long ago, two optimistic conjectures about the nature of Vassiliev invariants were widely circulated:

**Conjecture 1** *Vassiliev invariants distinguish all prime, unoriented knots.*

**Conjecture 2** [1] *All Vassiliev invariants are linear combinations of derivatives at  $q = 1$  of quantum Lie group invariants.*

Let  $s$  be a satellite operation on knots, so that if  $K$  is a knot,  $s(K)$  is a satellite of  $K$ . We say that an  $X$ -valued invariant  $V$  on knots *intertwines*  $s$  if  $V(s(K)) = s_X(V(K))$  for some  $s_X : X \rightarrow X$ , so that the invariant  $V \circ s$  yields no more information than  $V$  itself. In this paper, we present the following three theorems.

**Theorem 3** *Let  $V$  be a knot invariant which intertwines satellite operations. Then either  $V$  distinguishes all oriented knots, or there exist prime, unoriented knots which are not distinguished by  $V$ .*

More precisely, Theorem 3 states that, assuming  $V$  distinguishes prime, unoriented knots, it distinguishes any pair  $K_1$  and  $K_2$  of distinct oriented knots. Strictly speaking, we will only prove this when  $K_2$  is the inverse of  $K_1$  (i.e.,  $K_1$  with the opposite orientation), but this is in fact the main case. The alternative case where they are not inverses is settled as follows: Knot inversion is itself a satellite operation, so  $V(K_1) = V(K_2)$  implies  $V(-K_1) = V(-K_2)$ , which is to say that  $V$  does not distinguish  $K_1$  and  $K_2$  as unoriented knots. If they are not both prime, then by Lemma 7, they can be made prime by a satellite operation and they will remain indistinguishable.

**Theorem 4** *The universal Vassiliev invariant  $v_n$  of order  $n$  intertwines satellite operations.*

**Theorem 5** *The universal quantum link invariant for a Lie algebra  $\mathfrak{g}$  intertwines satellite operations.*

It is known that the quantum link invariants do not detect invertibility. Therefore, the three theorems have the following corollary.

**Corollary 6** *Conjectures 1 and 2 are mutually exclusive.*

Our arguments are inspired by the close relation between inverting a knot and the operation of mild mutation on a 3-manifold  $M$  such as a knot complement. Mild mutation consists of cutting  $M$  along a torus and then regluing after applying the central involution of the mapping class group. This operation was similarly exploited by Kania-Bartoszyńska to exhibit two closed 3-manifolds with the same quantum 3-manifold invariants [5].

Despite the pessimistic aspect of Theorem 3, it seems likely that quantum invariants, which for links are determined by Vassiliev invariants, distinguish all atoroidal, irreducible 3-manifolds, in particular unoriented knots which are not satellites.

## 1 Satellite operations

Let  $\mathcal{K}$  be the set of framed, oriented knots in  $S^3$  up to isotopy. Suppose that  $L$  is a connected, framed, oriented two-component link whose components are labelled as  $L_1$  and  $L_2$ . Suppose further that  $L_1$  is an un-knot with untwisted framing. Then a knot  $K \in \mathcal{K}$  together with  $L$  yield a satellite knot  $s_L(K)$  of  $K$  by removing a tube around  $K$  and a tube around  $L_1$  and gluing the two knot complements together to retain only  $L_2$ , specifically by gluing the longitude of  $K$  (given by the framing) to the meridian of  $L_1$  and vice-versa. In other words,  $L$  yields a satellite operation  $s_L : \mathcal{K} \rightarrow \mathcal{K}$ .

There are two ways in which  $s_L(K)$  might be a composite knot. Firstly, any summand of  $L$  (necessarily a summand of  $L_2$ ) appears as a summand of  $s_L(K)$ . Secondly, if  $K$  is trivial, then  $s_L(K)$  can be substantially simpler than and different from  $L$  and can be a composite knot, even if  $L$  is prime.

**Lemma 7** *If  $L$  is prime and  $K$  is non-trivial, then  $s_L(K)$  is prime, even if  $K$  is composite.*

This is a classical result that is proved by an innermost circle argument: Assume a sphere  $S$  that separates  $s_L(K)$  into summands and put it in a position of minimal transverse intersection with the boundary torus of  $K$ . An analysis of the innermost circles of intersection on  $S$  yields a summand of  $L$  or a new sphere with smaller intersection.

Now suppose that  $L$  is hyperbolic and does not admit a symmetry that inverts  $L_1$ . Let  $K$  be a non-invertible knot and let  $-K$  be its inverse. Then  $s_L(K)$  and  $s_L(-K)$  are prime, and they differ as unoriented knots as follows: Since  $L$  is hyperbolic, its complement is an aspherical, atoroidal, and acylindrical 3-manifold (*i.e.*,  $L$  is prime, connected, not the Hopf link, and not a non-trivial satellite). Let  $T$  be the torus jacketing  $s_L(K)$  (*i.e.*, the gluing torus in the construction of the satellite), and let  $h$  be a diffeomorphism of  $S^3$  that takes  $s_L(K)$  to  $s_L(-K)$  such that  $T$  and  $h(T)$  are in a position of minimal transverse intersection. If the intersection is non-empty, then by another standard analysis of innermost circles, the result is either an essential sphere, cylinder, or torus, or a smaller intersection [4]. If the intersection is empty, then  $T$  and  $h(T)$  must be parallel, for otherwise  $L$  would have an essential torus. But in this case,  $h$  must invert either the  $K$  piece or the  $L$  piece, a contradiction.

For an explicit example, we can take  $K$  to be the knot  $8_{17}$ , as shown in Figure 1, and  $L$  to be the link  $9_{34}^2$ , as shown in Figure 2, using Conway's enumeration [9]. According to SnapPea [11] and Mostow rigidity, neither  $K$  nor  $L$  admit the undesired symmetries. Indeed,  $L$  has no symmetries, and we can take either component to be  $L_1$ .

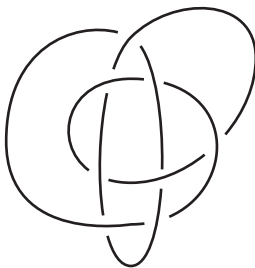


Figure 1: The non-invertible knot  $8_{17}$ .

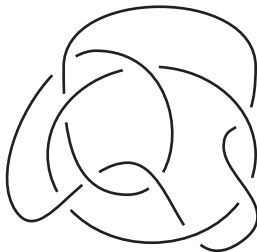


Figure 2: The link  $9^2_{34}$ , which has no symmetries.

Suppose that  $X$  is a set and  $f : \mathcal{K} \rightarrow X$  is an invariant which intertwines all satellite operations. We say that  $f$  *distinguishes*  $K_1$  and  $K_2$  as unoriented knots if the sets  $\{f(K_1), f(-K_1)\}$  and  $\{f(K_2), f(-K_2)\}$  differ. Let  $K$  and  $L$  be as above, let  $L'$  be  $L$  with  $L_2$  inverted, and suppose that  $f$  distinguishes  $s_L(K)$  and  $s_L(-K)$  as unoriented knots. Then it is immediate that either  $f \circ s_L$  or  $f \circ s_{L'}$  distinguishes  $K$  from  $-K$ . Since  $f$  intertwines these,  $f$  must also distinguish  $K$  from  $-K$ . This establishes Theorem 3.

It remains to show that  $v_n$ , the universal Vassiliev invariant of order  $n$ , and  $Q_{q, \mathfrak{g}}(K)$ , the universal quantum invariant for a knot  $K$  and a Lie algebra  $\mathfrak{g}$ , both intertwine all satellite operations.

## 2 Vassiliev invariants

As before, let  $\mathcal{K}_0 = \mathcal{K}$  be the set of framed, oriented knots in  $S^3$ , and let  $\mathcal{K}_n$  be the set of framed, oriented, immersed circles in  $S^3$  with  $n$  double points with transverse tangents. (The immersed circles will loosely be called generalized knots.) For example,  $\mathcal{K}$  and  $\mathcal{K}_1$  are the codimension 0 and codimension 1 cells of a stratification of the space of smooth maps from  $S^1$  to  $S^3$ . Let  $V$  be the free abelian group generated by the union of all  $\mathcal{K}_n$ 's, modulo the relation that

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \nearrow \searrow \end{array} - \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array}$$

for every triple of knots which differ at one crossing as indicated. Note that  $\mathcal{K}$  is a basis for  $V$ . The Vassiliev space  $V_n$  of order  $n$  is the quotient of  $V$  by the subgroup generated by  $\mathcal{K}_{n+1}$ , and the universal Vassiliev invariant of order  $n$  is the induced map  $v_n : \mathcal{K} \rightarrow V_n$ . Theorem 4 asserts that for any two-component link  $L$ ,  $v_n \circ s_L = s_{L, V_n} \circ v_n$  for some endomorphism  $s_{L, V_n}$  of  $V_n$ . Shifting  $n$  by 1, if  $J_n$  is the subspace (or ideal) of  $V$  generated by  $\mathcal{K}_n$ , then, equivalently, we wish to show that  $s_L(J_n) \subseteq J_n$ .

Let  $C$  be the free abelian group whose basis consists of a left-handed crossing  $l$  and a right-

handed crossing  $r$ , not part of any knot:

$$\begin{aligned} l &= \begin{array}{c} \nearrow \\ \searrow \end{array} \\ r &= \begin{array}{c} \nwarrow \\ \searrow \end{array} \end{aligned}$$

Let  $a = r - l \in C$ , and let  $A_{1,1}$  be the subgroup of  $C$  spanned by  $a$ . More generally, let  $A_{n,k} \subset C^{\otimes n}$  be the span of all subgroups of the form  $A_{1,1}^{\otimes k} \otimes C^{\otimes n-k}$  with the tensor factors permuted arbitrarily. Given a generalized knot  $K \in \mathcal{K}_n$  with linearly ordered double points (not necessarily ordered as they appear along  $K$ ), there is a map  $\phi_K : C^{\otimes n} \rightarrow V$  which acts on an element of the product basis by replacing each double point of  $K$  by the corresponding factor crossing. For example, if  $K$  is the generalized knot shown in Figure 3, then  $\phi_K(l \otimes r)$  is the knot shown in Figure 4. In particular,  $\phi_K(a^{\otimes n})$  is simply  $K$  itself as an element of  $V$ . More generally,  $\phi_K(A_{n,k}) \subseteq \mathcal{K}_k$ . Finally, let  $a_{n,\Delta} = r^{\otimes n} - l^{\otimes n} \in C^{\otimes n}$ .

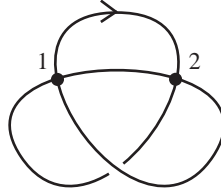


Figure 3: A generalized knot  $K$  with ordered double points.

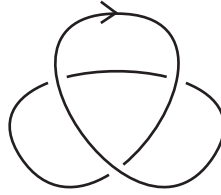


Figure 4: The knot  $\phi_K(l \otimes r)$ .

Let  $L$  be a 2-component link and let  $K$  be as before. We can take the satellite of  $K$ , of sorts, corresponding to  $L$  by replacing every double point by  $m^2$  double points, replacing every crossing by  $m^2$  crossings, every arc but one by  $m$  arcs, and the remaining arc by the pattern of  $L_2$ , as shown in Figure 5. The result  $K'$  is not unique; it depends on where we place the pattern of  $L_2$ , and if  $L$  is rearranged, even  $n$  can change. Nevertheless, the vector

$$b = \phi_{K'}(a_{m^2,\Delta}^{\otimes n})$$

in  $V$  is uniquely determined by  $K$  and  $L$  up to sign, because it equals  $\pm s_L(K)$ , where  $s_L$  is interpreted as an endomorphism of  $V$  and  $K$  is interpreted as an element of  $V$ . Here we order the double points of  $K'$  in bunches according to which double points of  $K$  they come from. (The sign is due to the fact that the satellite operations may replace a left-handed crossing by a combination of right- and left-handed crossings.) In other words,  $K$  represents a certain linear combination of honest knots in  $V$ , and  $s_L(K) = \pm b$  is the same linear combination of the  $L$ -satellites of these knots.

Observe that  $a_{m^2,\Delta} \in A_{m^2,1}$ . It follows that

$$a_{m^2,\Delta}^{\otimes n} \in A_{m^2,1}^{\otimes n} \subset A_{nm^2,n}.$$

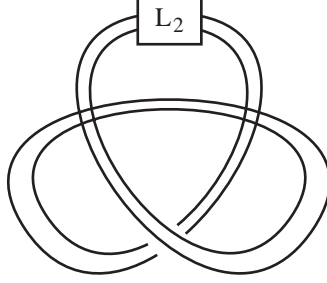


Figure 5: A fake satellite operation on generalized knots.

Since  $\phi_{K'}(A_{nm^2,n}) \subseteq J_n$ , we can conclude that  $b = s_*(K) \in J_n$  for every  $K \in \mathcal{K}_n$ . which is equivalent to the desired result that  $s_L(J_n) \subseteq J_n$ , which proves Theorem 4.

### 3 Quantum Lie group invariants

In this section, we argue Theorem 5. Let  $\mathfrak{g}$  be a simple Lie group. Following the well-known theory of quantum topological invariants [10],  $U_q(\mathfrak{g})$  is a quasitriangular Hopf algebra, and there corresponds a polynomial invariant of oriented, framed graphs in  $S^3$  whose edges are labelled by finite-dimensional representations of  $U_q(\mathfrak{g})$  and whose vertices are labelled by invariant tensors over the representations of the incident edges. In this theory, parallel edges labelled by representations  $V_1, \dots, V_n$  are equivalent to one edge labelled by  $V_1 \otimes \dots \otimes V_n$ , and reversing the orientation of an edge is equivalent to taking the dual of its representation. The invariants are also multiadditive under direct sums.

The standard theory can be generalized in a simple way for links so that each link component is labelled by a formal trace  $\tau$  on  $U_q(\mathfrak{g})$  such that  $\tau(ab) = \tau(ba)$ . If  $\tau$  is the character of a linear representation, then the generalized invariant is the same as before. The theory is then multilinear over formal traces, which form a vector space  $T_q(\mathfrak{g})$ . In particular, given a knot  $K$ , there is a universal invariant  $Q_{q,\mathfrak{g}}(K) \in T_q(\mathfrak{g})^*$ . Moreover, the characters of irreducible representations form a basis for  $T_q(\mathfrak{g})$  since  $U_q(\mathfrak{g})$  is semisimple for  $q$  an indeterminate, so the generalization is in this case nothing more than the linear completion of the invariants in which edges are labelled by true representations.

Given a 2-component link  $L$  as in the previous section, if we interpret the second component  $L_2$  as a knot in a solid torus and label it by a representation  $V$ , then in the quantum topological invariant theory it represents some equivariant endomorphism of  $V_1 \otimes \dots \otimes V_n$ , where each  $V_i$  is either  $V$  or  $V^*$  (see Figure 6). The tensor product  $V = V_1 \otimes \dots \otimes V_i$  has a direct sum decomposition

$$V = \bigoplus_{W \in \mathcal{R}} m_W W,$$

where  $\mathcal{R}$  a set with one representative of each irreducible representation and  $m_W$  is a non-negative integer. The map  $L_2$ , as an endomorphism of  $V$ , decomposes as

$$L_2 = \bigoplus L_W \otimes I_W,$$

where  $L_W$  is an  $m_W \times m_W$  matrix over the ground field and  $I_W$  is the identity on  $W$ . If  $\tau_{q,\mathfrak{g}}(K, V)$  is the quantum invariant of  $K$  colored by the representation  $V$ , then

$$\tau_{q,\mathfrak{g}}(s_L(K), V) = \sum \text{Tr}(s_W) \tau_{q,\mathfrak{g}}(K, W) \quad (1)$$

The left side is a satellited quantum group invariant of the knot  $K$ . The right side is a linear combination of ordinary quantum group invariants with different representations. Equation 1 establishes that  $\tau_{q,\mathfrak{g}}(s_L(K), V)$  factors through  $Q_{q,\mathfrak{g}}(K)$ . Therefore  $s_L$  intertwines  $Q_{q,\mathfrak{g}}$ .

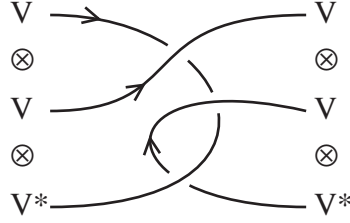


Figure 6: A tangle inducing a linear endomorphism.

That quantum group invariants are invariant under inversion is equivalent to the statement that

$$\tau_{q,\mathfrak{g}}(K, W) = \tau_{q,\mathfrak{g}}(K, W^*)$$

for every  $W$ . Recall that the quantum group  $U_q(\mathfrak{g})$  is constructed from the Dynkin diagram of  $\mathfrak{g}$ , and that any such Dynkin diagram has a dualizing automorphism, *i.e.*, an automorphism which takes the highest weight of an irreducible representation  $W$  to the highest weight of the dual representation  $W^*$ . It follows that the pair  $(U_q(\mathfrak{g}), W)$  is isomorphic to the pair  $(U_q(\mathfrak{g}), W^*)$ , and that the invariants are equal.

## 4 Finite group invariants

The treatment of quantum group invariants generalizes from  $U_q(\mathfrak{g})$  to arbitrary quasitriangular Hopf algebras  $H$ . However, in the general case, there is no guarantee that a dualizing homomorphism exists. In particular, if  $H = D(\mathbb{C}[G])$  is the quantum double of a finite group algebra, the corresponding invariant  $\tau_H$  counts group homomorphisms from  $\pi_1(S^3 - K)$  to  $G$  with a specified restriction to the peripheral subgroup. Strictly speaking, one fixes a group element  $g$  and a linear representation  $V$  of the centralizer of  $g$ , and one takes the sum of  $\text{Tr}_V(l)$  over homomorphisms that take  $m$  to  $g$ , where  $m$  is the meridian and  $l$  is the longitude. However, after a transformation, this is the same as counting homomorphisms that take  $(m, l)$  to  $(g, h)$  for fixed  $g$  and  $h$ . Although it is interesting to consider this invariant in quantum terms [2, 8], a more sober point of view in practice is that of classical group theory and algebraic topology.

Define a group-theoretic knot to be a triple  $(G, m, l)$ , where  $G$  is a group and  $m$  and  $l$  are two arbitrary commuting elements of  $G$  called the meridian and the longitude. An honest knot  $K$  in  $S^3$  yields a group-theoretic knot  $(\pi_1(S^3 - K), m, l)$  which determines  $K$  completely. Define  $(G, m, l)$  to be *invertible* if there exists an automorphism of  $G$  that inverts  $m$  and  $l$ . This too is equivalent to invertibility for honest knots. If the number of homomorphisms from  $(\pi_1(S^3 - K), m_K, l_K)$  to  $(G, m_G, l_G)$  differs from the number of homomorphisms to  $(G, m_G^{-1}, l_G^{-1})$ , then  $K$  is clearly non-invertible. Moreover, by inclusion-exclusion over subgroups of  $G$ , counting homomorphisms yields the same information as counting epimorphisms.

Let  $C_n$  be the  $n$ th commutator subgroup of  $\pi = \pi_1(S^3 - K)$ . The first interesting group-theoretic model of  $K$  is  $\pi/C_2$ , whose structure is characterized by the Alexander polynomial  $\Delta_K(t)$ . Unfortunately, the well-known identity  $\Delta_K(t) = \Delta_K(t^{-1})$  implies that  $\pi/C_2$  is an invertible group-theoretic knot. In other words, all 3-dimensional knots are invertible at the metabelian level; if  $G$  is metabelian, counting homomorphisms to  $G$  cannot detect invertibility.

The quotient  $\pi/C_3$  is more fruitful: Hartley [3] found that the homology of metabelian coverings often does detect invertibility, which amounts to counting homomorphisms to certain meta-metabelian groups (groups with metabelian commutator). However, neither these groups nor any other solvable groups will yield any information when the Alexander polynomial of  $K$  is trivial. In this case,  $C_1$  is a perfect group, which means that  $\pi/C_n \cong \mathbb{Z}$  for all  $n$ , which is the same as the unknot, which is invertible. The first example of such a knot is Conway's 11-crossing knot, shown in Figure 7.

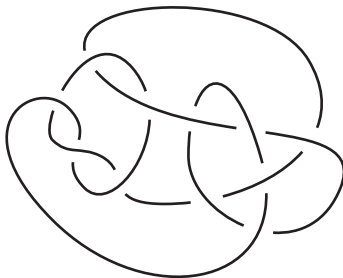


Figure 7: Conway's knot, which has trivial Alexander polynomial.

The sporadic simple group  $M_{11}$  of order 7920 admits no automorphism that inverts an element  $g$  of order 11. We report here that, if  $K$  is the Conway knot, then  $\pi_1(S^3 - K)$  admits precisely one epimorphism to  $M_{11}$  that sends the meridian  $m$  to either  $g$  or  $g^{-1}$ . It follows that  $K$  is non-invertible. This result was obtained by a computer search: working with the Wirtinger presentation of the knot group, the algorithm finds among all maps from the generating set to the conjugacy class of  $g$  those which satisfy the relations. Note that Wirtinger generators are conjugate because they are all meridians, so we need not consider other maps from the generators to  $M_{11}$ . The first generator can be mapped to  $g$  itself, and if the generators are properly ordered, the first three determine the others in the case of the Conway knot. Those who wish to investigate the validity of this computational result can find the author's software on the Internet [7].

SnapPea can also prove that  $K$  is non-invertible using the hyperbolic structure of  $S^3 - K$  and Mostow rigidity, but the proof using  $M_{11}$  is perhaps more elementary, although it does involve computer calculations.

Kenichi Kawagoe [6] has verified the author's computational results, and has also verified by computer that the finite groups  $Sz(8)$  and  $\text{Aut}(U_3(3))$  can be used to determine the non-invertibility of both the Conway knot and the Kinoshita-Terasaka knot, which is a mutant of the Conway knot. These results support the hypothesis that a knot group, like a free group, has a good chance of admitting any particular finite group  $G$  as a quotient, provided that  $G$  is generated by sufficiently few elements. While such a quotient may prove useful as an invariant, it does not necessarily reveal any special connection between the knot and the group  $G$ .

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